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LETTER TO THE EDITOR

Clump lifetime of relative motion in a stochastic force field

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Abstract. A Fokker–Planck equation describing the relative motion of two particles in a dissipative spacetime correlated turbulent force-field flow is derived, and the clump lifetime of the correlated motion for two adjacent particles with a small initial separation is found and compared with Dupree's results for the dissipationless case.

The study of the relative motion of particles in a spacetime correlated turbulent flow has been of continuous interest in the literature [1–5]. In particular, Dupree [1] and Inaba and Suzuki [5] have observed that two particles, which are initially adjacent to each other in phase space, are effectively separated from each other in a time interval of the order of the so-called clump lifetime

$$\tau_{\mathrm{cl}} = (4\chi^2 Q)^{-1/3} \ln(1/\chi^2 \xi_0^2)$$

where χ denotes the characteristic wavenumber of the turbulent force-field flow, Q is the diffusion coefficient, and ξ_0 is the initial separation between two adjacent particles. The above clump lifetime was obtained in the highly inertial regime, where dissipation could be ignored.

In this letter, the diffusive relative motion in spacetime turbulent fields is investigated for a dissipative case. We investigate in what manner can the presence of a dissipative mechanism modify the clump lifetime and the overall long-time behaviour of the system. To this end, we derive an exact Fokker–Planck equation describing the relative motion of two particles in a dissipative spacetime correlated turbulent force-field flow. The theoretical analysis is carried out using the formalism of functional calculus [6, 7]. A generalised Dupree's equation for the mean-squared relative displacement is then derived from the Fokker–Planck equation obtained. For a small initial separation, the clump lifetime of the correlated motion for two adjacent particles is found and compared with Dupree's results for the dissipationless case.

Consider the equations of motion for two particles ($i = 1, 2$) in a stochastic force field described by

$$\dot{\boldsymbol{X}}_i(t) = \boldsymbol{V}_i(t) \tag{1}$$

$$\dot{\boldsymbol{V}}_i(t) + (\nu/m)\boldsymbol{V}_i(t) = \boldsymbol{F}(\boldsymbol{X}_i(t), t)/m. \tag{2}$$

The turbulent field $\boldsymbol{F}(\boldsymbol{x}, t)$ is assumed to be a Gaussian stochastic process with zero-mean value

$$\langle \boldsymbol{F}(\boldsymbol{x}, t)\rangle = 0 \tag{3}$$

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and its spacetime correlation is given by [5]

$$\langle \boldsymbol{F}_\alpha(\boldsymbol{x}, t)\boldsymbol{F}_\beta(\boldsymbol{x}', t')\rangle = \delta_{\alpha\beta}Q(|\boldsymbol{x}-\boldsymbol{x}'|)\delta(t-t'). \tag{4}$$

We introduce the probability density with respect to the position and velocity of the two particles [2, 5]

$$P(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2; t) = \left\langle \prod_{i=1}^{2} \delta(\boldsymbol{x}_i - \boldsymbol{X}_i(t))\delta(\boldsymbol{v}_i - \boldsymbol{V}_i(t)) \right\rangle \tag{5}$$

where $\boldsymbol{X}_i(t)$ and $\boldsymbol{V}_i(t)$ are the solutions to (1) and (2), namely

$$\boldsymbol{X}_i(t) = \boldsymbol{X}_i(0) + \int_0^t \boldsymbol{V}_i(t')\,\mathrm{d}t' \tag{6}$$

and

$$\boldsymbol{V}_i(t) = \mathrm{e}^{-(\nu t/m)}\left\{ \boldsymbol{V}_i(0) + \frac{1}{m}\int_0^t \mathrm{e}^{(\nu t'/m)}\boldsymbol{F}[\boldsymbol{X}_i(t'), t']\,\mathrm{d}t' \right\} \tag{7}$$

from which we obtain

$$\frac{\delta \boldsymbol{V}_i(t)}{\delta \boldsymbol{F}_j(z, t')} = \frac{1}{m}\mathrm{e}^{-\nu(t-t')/m}\delta_{ij}\delta(z - \boldsymbol{X}_i(t'))\Theta(t-t') \tag{8}$$

where $\Theta(t-t')$ is the Heaviside function.

By differentiating (5) with respect to t and by taking into account (1) and (2), we obtain

$$\begin{aligned}\frac{\partial P}{\partial t} &= \left\langle \frac{\partial}{\partial t}\left(\prod_{i=1}^{2} \delta(\boldsymbol{x}_i - \boldsymbol{X}_i(t))\delta(\boldsymbol{v}_i - \boldsymbol{V}_i(t)) \right)\right\rangle \\ &= \sum_{j=1}^{2} \left\{ -\boldsymbol{v}_j \frac{\partial P}{\partial \boldsymbol{x}_j} + \frac{\nu}{m}\frac{\partial}{\partial \boldsymbol{v}_j}(\boldsymbol{v}_j P) \right. \\ &\quad \left. - \frac{1}{m}\frac{\partial}{\partial \boldsymbol{v}_j}\left\langle \boldsymbol{F}(\boldsymbol{x}_j, t)\prod_{i=1}^{2} \delta(\boldsymbol{x}_i - \boldsymbol{X}_i(t))\delta(\boldsymbol{v}_i - \boldsymbol{V}_i(t)) \right\rangle \right\}. \end{aligned} \tag{9}$$

With the help of the Furutsu-Novikov formula [2, 6, 7]

$$\langle \boldsymbol{F}(\boldsymbol{x}, t)R[F]\rangle = \iint \mathrm{d}y\,\mathrm{d}t' \langle \boldsymbol{F}(\boldsymbol{x}, t)\boldsymbol{F}(y, t')\rangle\langle \delta R[F]/\delta \boldsymbol{F}(y, t')\rangle \tag{10}$$

we arrive at

$$\frac{\partial P}{\partial t} = \sum_{i,j=1}^{2} \left\{ -\boldsymbol{v}_j \frac{\partial P}{\partial \boldsymbol{x}_j} + \frac{\nu}{m}\frac{\partial}{\partial \boldsymbol{v}_j}(\boldsymbol{v}_j P) + \frac{1}{m^2}\frac{\partial^2}{\partial \boldsymbol{v}_j\,\partial \boldsymbol{v}_i}[Q(|\boldsymbol{x}_i - \boldsymbol{x}_j|)P] \right\}. \tag{11}$$

Now, we can reduce (11) to an equation with respect to the relative position and velocity, $\boldsymbol{\xi} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{\mu} = \boldsymbol{v}_1 - \boldsymbol{v}_2$,

$$\frac{\partial P}{\partial t} = -\boldsymbol{\mu}\frac{\partial P}{\partial \boldsymbol{\xi}} + \frac{\nu}{m}\frac{\partial}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}P) + \frac{2}{m^2}[Q(0) - Q(\boldsymbol{\xi})]\frac{\partial^2 P}{\partial \boldsymbol{\mu}^2}. \tag{12}$$

From (12), we obtain the coupled moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{\xi}^2(t)\rangle = 2\langle \boldsymbol{\xi\mu}\rangle \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{\xi\mu}\rangle = \langle \boldsymbol{\mu}^2\rangle - \frac{\nu}{m}\langle \boldsymbol{\xi\mu}\rangle \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{\mu}^2\rangle = -\frac{2\nu}{m}\langle \boldsymbol{\mu}^2\rangle + \frac{4}{m^2}\langle Q(0) - Q(\boldsymbol{\xi})\rangle \tag{15}$$

which yield the equation for the mean-squared relative displacement

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}t^3} + \frac{3\nu}{m}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{2\nu^2}{m^2}\frac{\mathrm{d}}{\mathrm{d}t}\right)\langle \boldsymbol{\xi}^2(t)\rangle = \frac{8}{m}\langle Q(0) - Q(\boldsymbol{\xi})\rangle. \tag{16}$$

If $P(\boldsymbol{\xi}, \boldsymbol{\mu}; t)$ has a significant value only around $\boldsymbol{\xi} = 0$ [5], one can make the approximation

$$\langle Q(\boldsymbol{\xi})\rangle \simeq Q(0)[1 - \chi^2\langle \boldsymbol{\xi}^2\rangle] \tag{17}$$

whereby (16) is reduced to

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}t^3} + \frac{3\nu}{m}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{2\nu^2}{m^2}\frac{\mathrm{d}}{\mathrm{d}t}\right)\langle \boldsymbol{\xi}^2(t)\rangle = \frac{8}{m}Q(0)\chi^2\langle \boldsymbol{\xi}^2(t)\rangle \tag{18}$$

which is a generalisation of Dupree's equation [1, 5].

If the initial separation of two adjacent particles is small ($\xi_0 \ll \chi^{-1}$), their relative distance grows exponentially. Now, the clump lifetime, defined as the time when the relative distance between the two particles just exceeds χ^{-1}, is given by

$$\tau_{\mathrm{cl}} = \tau_0 \ln(1/\chi^2\xi_0^2) \tag{19}$$

where the new time constant τ_0 is obtained from the cubic algebraic equation

$$(\tau_0^{-1})^3 + (3\nu/m)(\tau_0^{-1})^2 + (2\nu^2/m^2)(\tau_0^{-1}) - (8/m)Q(0)\chi^2 = 0. \tag{20}$$

By first defining

$$\tau_0^* = \tau_0[(8/m)Q(0)\chi^2]^{1/3} \tag{21}$$

$$\nu^* = \nu[(8/m)Q(0)\chi^2]^{1/3} \tag{22}$$

we find that:

(*a*) for $\nu^* \leqslant [27/4]^{1/6}$

$$(\tau_0^*)^{-1} = \left\{\frac{1}{2} + \left[\frac{1}{4} - \frac{(\nu^*)^6}{27}\right]^{1/2}\right\}^{1/3} + \left\{\frac{1}{2} - \left[\frac{1}{4} - \frac{(\nu^*)^6}{27}\right]^{1/2}\right\}^{1/3} - \nu^* \tag{23}$$

(*b*) for $\nu^* > [27/4]^{1/6}$

$$(\tau_0^*)^{-1} = \frac{2\nu^*}{\sqrt{3}}\cos\left\{\frac{1}{3}\tan^{-1}\left[\frac{4(\nu^*)^6}{27} - 1\right]^{1/2}\right\} - \nu^* \tag{24}$$

such that when $\nu^* \to 0$, $\tau_0^* \to 1$ (it reduces to Dupree's result), and when $\nu^* \to \infty$, $\tau_0^* \to \infty$.

For long times, $\langle Q(\boldsymbol{\xi})\rangle$ is assumed to be a small quantity and can be neglected in (16) [5]. It then follows that a *discontinuity* exists in the asymptotic behaviour of the solution of the above equation: for *any finite* amount of dissipation (no matter how small) the system exhibits diffusivity $\langle \boldsymbol{\xi}^2(t)\rangle \to (4/3m)Q(0)t$ ($t \to \infty$), while Richardson's third-power law ($\langle \boldsymbol{\xi}^2(t)\rangle \to t^3$) is recovered only if the dissipation is set *exactly* to zero from the outset.

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